

# Exponential stability of the wave equation with memory and time delay

F. ALABAU-BOUSSOUIRA

LMAM, Université de Lorraine and CNRS (UMR 7122)  
57045 Metz Cedex 1, France

S. NICAISE

LAMAV, FR CNRS 2956, Institut des Sciences et Techniques de Valenciennes  
Université de Valenciennes et du Hainaut Cambrésis  
59313 Valenciennes Cedex 9, France

C. PIGNOTTI

Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica  
Università di L'Aquila, 67010 L'Aquila, Italy

## Abstract

We study the asymptotic behaviour of the wave equation with viscoelastic damping in presence of a time-delayed damping. We prove exponential stability if the amplitude of the time delay term is small enough.

## 1 Introduction

This paper is devoted to the stability analysis of a viscoelastic model. In particular, we consider a model combining viscoelastic damping and time-delayed damping. We prove an exponential stability result provided that the amplitude of time-delayed damping is small enough. Moreover, we give a precise estimate on this smallness condition. This shows that even if delay effects usually generate instability (see e.g. [5, 6, 12, 18]), the damping due to viscoelasticity can counterbalance them.

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded set with a smooth boundary. Let us consider the following problem:

$$u_{tt}(x, t) - \Delta u(x, t) - \int_0^\infty \mu(s) \Delta u(x, t - s) ds + ku_t(x, t - \tau) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (1.1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (1.2)$$

$$u(x, t) = u_0(x, t) \quad \text{in } \Omega \times (-\infty, 0] \quad (1.3)$$

where the initial datum  $u_0$  belongs to a suitable space, the constant  $\tau > 0$  is the time delay,  $k$  is a real number and the memory kernel  $\mu : [0, +\infty) \rightarrow [0, +\infty)$  is a locally absolutely continuous function satisfying

- i)  $\mu(0) = \mu_0 > 0$ ;
- ii)  $\int_0^{+\infty} \mu(t) dt = \tilde{\mu} < 1$ ;
- iii)  $\mu'(t) \leq -\alpha\mu(t)$ , for some  $\alpha > 0$ .

We know that the above problem is exponentially stable for  $k = 0$  (see e.g. [8]).

We will show that an exponential stability result holds if the delay parameter  $k$  is *small* with respect to the memory kernel.

Observe that for  $\tau = 0$  and  $k > 0$  the model (1.1) – (1.3) presents both viscoelastic and standard dissipative damping. Therefore, in that case, under the above assumptions on the kernel  $\mu$ , the model is exponentially stable.

We will see that exponential stability also occurs for  $k < 0$ , under a suitable smallness assumption on  $|k|$ . Note that the term  $ku_t(t)$  with  $k < 0$  is a so-called anti-damping (see e.g. [7]), namely a damping with an opposite sign with respect to the standard dissipative one, and therefore it induces instability. Indeed, in absence of viscoelastic damping, i.e. for  $\mu \equiv 0$ , the solutions of the above problem, with  $\tau = 0$  and  $k < 0$ , grow exponentially to infinity.

We will prove our stability results by using a perturbative approach, first introduced in [16] (see also [13] for a more general setting).

The stabilization problem for model (1.1) – (1.3) has been studied also by Guesmia in [9] by using a different approach based on the construction of a suitable Lyapunov functional. Our analysis allows to determine an explicit estimate on the constant  $k_0$  (cf. Theorem 2.2). Moreover, our approach can be extended to the case of localized viscoelastic damping (cf. [11]). In fact, we first prove the exponential stability of an auxiliary problem having a decreasing energy and then, regarding the original problem as a perturbation of that one, we extend the exponential decay estimate to it.

The paper is organized as follows. In sect. 2 we study the well-posedness by introducing an appropriate functional setting and we formulate our stability result. In sect. 3 we introduce the auxiliary problem and prove the exponential decay estimate for it. Then, the stability result is extended to the original problem.

## 2 Main results and preliminaries

As in [4], let us introduce the new variable

$$\eta^t(x, s) := u(x, t) - u(x, t - s). \quad (2.1)$$

Moreover, as in [12], we define

$$z(x, \rho, t) := u_t(x, t - \tau\rho), \quad x \in \Omega, \quad \rho \in (0, 1), \quad t > 0. \quad (2.2)$$

Using (2.1) and (2.2) we can rewrite (1.1)–(1.3) as

$$\begin{aligned} u_{tt}(x, t) &= (1 - \tilde{\mu})\Delta u(x, t) + \int_0^\infty \mu(s)\Delta\eta^t(x, s)ds \\ &\quad - kz(x, 1, t) \quad \text{in } \Omega \times (0, +\infty) \end{aligned} \quad (2.3)$$

$$\eta_t^t(x, s) = -\eta_s^t(x, s) + u_t(x, t) \quad \text{in } \Omega \times (0, +\infty) \times (0, +\infty), \quad (2.4)$$

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 \quad \text{in } \Omega \times (0, 1) \times (0, +\infty), \quad (2.5)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (2.6)$$

$$\eta^t(x, s) = 0 \quad \text{in } \partial\Omega \times (0, +\infty), \quad t \geq 0, \quad (2.7)$$

$$z(x, 0, t) = u_t(x, t) \quad \text{in } \Omega \times (0, +\infty), \quad (2.8)$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \quad (2.9)$$

$$\eta^0(x, s) = \eta_0(x, s) \quad \text{in } \partial\Omega \times (0, +\infty), \quad (2.10)$$

$$z(x, \rho, 0) = z^0(x, -\tau\rho) \quad x \in \Omega, \quad \rho \in (0, 1), \quad (2.11)$$

where

$$\begin{aligned} u_0(x) &= u_0(x, 0), \quad x \in \Omega, \\ u_1(x) &= \frac{\partial u_0}{\partial t}(x, t)|_{t=0}, \quad x \in \Omega, \\ \eta_0(x, s) &= u_0(x, 0) - u_0(x, -s), \quad x \in \Omega, \quad s \in (0, +\infty), \\ z^0(x, s) &= \frac{\partial u_0}{\partial t}(x, s), \quad x \in \Omega, \quad s \in (-\tau, 0). \end{aligned} \quad (2.12)$$

Let us denote  $\mathcal{U} := (u, u_t, \eta^t, z)^T$ . Then we can rewrite problem (2.3)–(2.11) in the abstract form

$$\begin{cases} \mathcal{U}' = \mathcal{A}\mathcal{U}, \\ \mathcal{U}(0) = (u_0, u_1, \eta_0, z^0)^T, \end{cases} \quad (2.13)$$

where the operator  $\mathcal{A}$  is defined by

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} := \begin{pmatrix} v \\ (1 - \tilde{\mu})\Delta u + \int_0^\infty \mu(s)\Delta w(s)ds - kz(\cdot, 1) \\ -w_s + v \\ -\tau^{-1}z_\rho \end{pmatrix}, \quad (2.14)$$

with domain

$$\begin{aligned} \mathcal{D}(\mathcal{A}) := \{ & (u, v, \eta, z)^T \in H_0^1(\Omega) \times H_0^1(\Omega) \times L_\mu^2((0, +\infty); H_0^1(\Omega)) \times H^1((0, 1); L^2(\Omega)) : \\ & v = z(\cdot, 0), \quad (1 - \tilde{\mu})u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \\ & \eta_s \in L_\mu^2((0, +\infty); H_0^1(\Omega)) \}, \end{aligned} \quad (2.15)$$

where  $L_\mu^2((0, \infty); H_0^1(\Omega))$  is the Hilbert space of  $H_0^1$ -valued functions on  $(0, +\infty)$ , endowed with the inner product

$$\langle \varphi, \psi \rangle_{L_\mu^2((0, \infty); H_0^1(\Omega))} = \int_\Omega \left( \int_0^\infty \mu(s) \nabla \varphi(x, s) \nabla \psi(x, s) ds \right) dx.$$

Denote by  $\mathcal{H}$  the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L_\mu^2((0, \infty); H_0^1(\Omega)) \times L^2((0, 1); L^2(\Omega)),$$

equipped with the inner product

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} &:= (1 - \tilde{\mu}) \int_\Omega \nabla u \nabla \tilde{u} dx + \int_\Omega v \tilde{v} dx + \int_\Omega \int_0^\infty \mu(s) \nabla w \nabla \tilde{w} ds dx \\ &+ \int_0^1 \int_\Omega z(x, \rho) \tilde{z}(x, \rho) dx d\rho. \end{aligned} \quad (2.16)$$

Combining the ideas from [17] with the ones from [12] (see also [3]), we can prove that the operator  $\mathcal{A}$  generates a strongly continuous semigroup ( $\mathcal{A} - cI$  is dissipative for a sufficiently large constant  $c > 0$ ) and therefore the next existence result holds.

**Proposition 2.1** *For any initial datum  $\mathcal{U}_0 \in \mathcal{H}$  there exists a unique solution  $\mathcal{U} \in C([0, +\infty), \mathcal{H})$  of problem (2.13). Moreover, if  $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A})$ , then*

$$\mathcal{U} \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

Let us define the energy  $F$  of problem (1.1) – (1.3) as

$$\begin{aligned} F(t) = F(u, t) &:= \frac{1}{2} \int_\Omega u_t^2(x, t) dx + \frac{1 - \tilde{\mu}}{2} \int_\Omega |\nabla u(x, t)|^2 dx \\ &+ \frac{1}{2} \int_0^{+\infty} \int_\Omega \mu(s) |\nabla \eta^t(s)|^2 ds dx + \frac{\theta |k| e^\tau}{2} \int_{t-\tau}^t e^{-(t-s)} \int_\Omega u_t^2(x, s) ds dx, \end{aligned} \quad (2.17)$$

where  $\theta$  is any real constant satisfying

$$\theta > 1. \quad (2.18)$$

We will prove the following exponential stability result.

**Theorem 2.2** *For any  $\theta > 1$  in the definition (2.17), there exists a positive constant  $k_0$  such that for  $k$  satisfying  $|k| < k_0$  there is  $\sigma > 0$  such that*

$$F(t) \leq F(0)e^{1-\sigma t}, \quad t \geq 0; \quad (2.19)$$

*for every solution of problem (1.1) – (1.3). The constant  $k_0$  depends only on the kernel  $\mu(\cdot)$  of the memory term, on the time delay  $\tau$  and on the domain  $\Omega$ .*

To prove our stability result we will make use of the following result of Pazy (Theorem 1.1 in Ch. 3 of [15]).

**Theorem 2.3** *Let  $X$  be a Banach space and let  $A$  be the infinitesimal generator of a  $C_0$  semigroup  $T(t)$  on  $X$ , satisfying  $\|T(t)\| \leq Me^{\omega t}$ . If  $B$  is a bounded linear operator on  $X$  then  $A + B$  is the infinitesimal generator of a  $C_0$  semigroup  $S(t)$  on  $X$ , satisfying  $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$ .*

Moreover, we will use the following lemma (see Th. 8.1 of [10]).

**Lemma 2.4** *Let  $V(\cdot)$  be a non negative decreasing function defined on  $[0, +\infty)$ . If*

$$\int_S^{+\infty} V(t)dt \leq CV(S) \quad \forall S > 0,$$

*for some constants  $C > 0$ , then*

$$V(t) \leq V(0) \exp\left(1 - \frac{t}{C}\right), \quad \forall t \geq 0.$$

**Remark 2.5** Observe that the well-posedness result in the case  $\tau = 0$ , namely viscoelastic wave equation with standard frictional damping or anti-damping, directly follows from Theorem 2.3. Furthermore, from Theorem 2.3 we can also deduce an exponential stability estimate under a suitable smallness assumption on  $|k|$ . Indeed, for  $|k|$  small, we can look at problem (1.1)–(1.3)(with  $\tau = 0$ ) as a perturbation of the wave equation with only the viscoelastic damping. And it is by now well-known that for the last model an exponential decay estimate is available (see e.g. [8]).

### 3 Stability results

In this section we will prove Theorem 2.2.

In order to study the stability properties of problem (1.1)–(1.3), we look at an auxiliary problem (cf. [16]) which is near to this one and more easier to deal with. Then, let us consider the system

$$u_{tt}(x, t) - \Delta u(x, t) + \int_0^\infty \mu(s) \Delta u(x, t-s) ds + \theta |k| e^\tau u_t(x, t) + k u_t(x, t-\tau) = 0 \quad \text{in } \Omega \times (0, +\infty) \quad (3.1)$$

$$u(x, t) = 0 \quad \text{on } \partial\Omega \times (0, +\infty) \quad (3.2)$$

$$u(x, t) = u_0(x, t) \quad \text{in } \Omega \times (-\infty, 0]. \quad (3.3)$$

First of all we show that the energy, defined by (2.17), of any solution of the auxiliary problem is not increasing.

**Proposition 3.1** *For every solution of problem (3.1) – (3.3) the energy  $F(\cdot)$  is not increasing and the following estimate holds*

$$\begin{aligned} F'(t) \leq & \frac{1}{2} \int_0^\infty \int_\Omega \mu'(s) |\nabla \eta^t(x, s)|^2 dx ds \\ & - \frac{|k|(\theta e^\tau - 1)}{2} \int_\Omega u_t^2(x, t) dx - \frac{|k|(\theta - 1)}{2} \int_\Omega u_t^2(x, t-\tau) dx \\ & - \frac{\theta |k| e^\tau}{2} \int_{t-\tau}^t e^{-(t-s)} \int_\Omega u_t^2(x, s) dx ds. \end{aligned} \quad (3.4)$$

**Remark 3.2** Note that the energy  $F(\cdot)$  of solutions of the original problem (1.1) – (1.3) is not in general decreasing.

**Proof of Proposition 3.1.** Differentiating (2.17) we have

$$\begin{aligned} F'(t) = & \int_\Omega u_t(x, t) u_{tt}(x, t) dx + (1 - \tilde{\mu}) \int_\Omega \nabla u(x, t) \nabla u_t(x, t) dx \\ & + \int_0^\infty \int_\Omega \mu(s) \nabla \eta^t(x, s) \nabla \eta_t^t(x, s) dx ds + \frac{\theta |k| e^\tau}{2} \int_\Omega u_t^2(x, t) dx \\ & - \frac{\theta |k|}{2} \int_\Omega u_t^2(x, t-\tau) dx - \frac{\theta |k| e^\tau}{2} \int_{t-\tau}^t e^{-(t-s)} \int_\Omega u_t^2(x, s) dx ds. \end{aligned}$$

Then, integrating by parts and using (2.4) and the boundary condition (3.2),

$$\begin{aligned} F'(t) = & \int_\Omega u_t(x, t) [u_{tt}(x, t) - (1 - \tilde{\mu}) \Delta u(x, t)] dx \\ & + \int_0^\infty \int_\Omega \mu(s) \nabla \eta^t(x, s) (\nabla u_t(x, t) - \nabla \eta_s^t(x, s)) dx ds + \frac{\theta |k| e^\tau}{2} \int_\Omega u_t^2(x, t) dx \\ & - \frac{\theta |k|}{2} \int_\Omega u_t^2(x, t-\tau) dx - \frac{\theta |k| e^\tau}{2} \int_{t-\tau}^t e^{-(t-s)} \int_\Omega u_t^2(x, s) dx ds. \end{aligned}$$

By using equations (3.1), (3.2), after integration by parts, we deduce

$$\begin{aligned}
F'(t) &= \int_{\Omega} u_t(t) \left[ - \int_0^{\infty} \mu(s) \Delta u(x, t-s) + \tilde{\mu} \Delta u(x, t) \right. \\
&\quad \left. - \theta |k| e^{\tau} u_t(x, t) - k u_t(x, t-\tau) \right] dx \\
&+ \int_0^{\infty} \int_{\Omega} \mu(s) \nabla \eta^t(x, s) \nabla u_t(x, t) dx ds + \frac{1}{2} \int_0^{\infty} \int_{\Omega} \mu'(s) |\nabla \eta^t(x, s)|^2 dx ds \\
&+ \frac{\theta |k| e^{\tau}}{2} \int_{\Omega} u_t^2(x, t) dx - \frac{\theta |k|}{2} \int_{\Omega} u_t^2(x, t-\tau) dx - \frac{\theta |k| e^{\tau}}{2} \int_{t-\tau}^t e^{-(t-s)} \int_{\Omega} u_t^2(x, s) dx ds \\
&= -\theta |k| e^{\tau} \int_{\Omega} u_t^2(x, t) dx - k \int_{\Omega} u_t(x, t) u_t(x, t-\tau) dx + \frac{\theta |k| e^{\tau}}{2} \int_{\Omega} u_t^2(x, t) dx \\
&\quad - \frac{\theta |k|}{2} \int_{\Omega} u_t^2(x, t-\tau) dx + \frac{1}{2} \int_0^{\infty} \int_{\Omega} \mu'(s) |\nabla \eta^t(x, s)|^2 dx ds \\
&\quad - \frac{\theta |k| e^{\tau}}{2} \int_{t-\tau}^t e^{-(t-s)} \int_{\Omega} u_t^2(x, s) dx ds.
\end{aligned}$$

Now, using Cauchy-Schwarz inequality we obtain (3.4). ■

**Corollary 3.3** *For every solution of problem (3.1) – (3.3), we have*

$$-\frac{1}{2} \int_S^T \int_0^{\infty} \int_{\Omega} \mu'(s) |\nabla \eta^t(x, s)|^2 dx ds \leq F(S), \quad (3.5)$$

and then by the condition  $\mu'(t) \leq -\alpha \mu(t)$  we directly get

$$\frac{1}{2} \int_S^T \int_0^{\infty} \mu(s) \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx ds dt \leq \frac{1}{\alpha} F(S). \quad (3.6)$$

**Proof.** As each term of the right-hand side of (3.4) is non positive, we directly get that

$$-\frac{1}{2} \int_S^T \int_0^{\infty} \int_{\Omega} \mu'(s) |\nabla \eta^t(x, s)|^2 dx ds \leq \int_S^T (-F'(t)) dt \leq F(S). \quad \blacksquare$$

**Theorem 3.4** *For any  $\theta > 1$  in the definition (2.17), there exist positive constants  $C$  and  $\bar{k}$ , depending on  $\mu$ ,  $\Omega$  and  $\tau$ , such that if  $|k| < \bar{k}$  then for any solution of problem (3.1) – (3.3) the following estimate holds*

$$\int_S^{+\infty} F(t) dt \leq C F(S) \quad \forall S > 0. \quad (3.7)$$

In order to prove Theorem 3.4 we need some preliminary results. Our proof relies in many points on [1] but we have to perform all computations because, in order to extend the exponential estimate related to the perturbed problem (3.1)–(3.3) to the original problem (1.1)–(1.3) we need to determine carefully all involved constants. From the definition of the energy we deduce

$$\begin{aligned}
\int_S^T F(t)dt &= \frac{1}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + \frac{1 - \tilde{\mu}}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt \\
&\quad + \frac{1}{2} \int_S^T \int_0^{\infty} \int_{\Omega} \mu(s) |\nabla \eta^t(x, s)|^2 dx ds dt \\
&\quad + \frac{\theta |k| e^{\tau}}{2} \int_S^T \int_{t-\tau}^t e^{-(t-s)} \int_{\Omega} u_t^2(x, s) ds dx dt.
\end{aligned} \tag{3.8}$$

Now, as in [1] we will use multiplier arguments in order to bound the right-hand side of (3.8). We note that we could not apply the same arguments directly to our original problem since the energy is not decreasing.

In the following we will denote by  $C_P$  the Poincaré constant, namely the smallest positive constant such that

$$\int_{\Omega} w^2(x) dx \leq C_P \int_{\Omega} |\nabla w(x)|^2 dx, \quad \forall w \in H_0^1(\Omega). \tag{3.9}$$

**Lemma 3.5** *Assume*

$$|k| < \frac{1 - \tilde{\mu}}{2C_P(\theta e^{\tau} + 1)}. \tag{3.10}$$

*Then, for any  $T \geq S \geq 0$  we have*

$$(1 - \tilde{\mu}) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt \leq C_0 \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + C_1 F(S), \tag{3.11}$$

*with*

$$C_0 = 2 + \theta |k| e^{\tau}, \quad C_1 = 4 \left( 1 + \frac{\tilde{\mu}}{\alpha(1 - \tilde{\mu})} + \frac{C_P}{1 - \tilde{\mu}} + \frac{1}{2(\theta - 1)} \right). \tag{3.12}$$

**Proof.** Multiplying equation (3.1) by  $u$  and integrating on  $\Omega \times [S, T]$  we have

$$\begin{aligned}
\int_S^T \int_{\Omega} [u_{tt}(x, t) - \Delta u(x, t) + \int_0^{\infty} \mu(s) \Delta u(x, t - s) ds \\
+ \theta |k| e^{\tau} u_t(x, t) + k u_t(x, t - \tau)] u(x, t) dx dt = 0.
\end{aligned}$$

So, integrating by parts and using the boundary condition (3.2), we get

$$\begin{aligned}
& - \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \left[ \int_{\Omega} u(x, t) u_t(x, t) dx \right]_S^T \\
& + \theta |k| e^{\tau} \int_S^T \int_{\Omega} u(x, t) u_t(x, t) dx dt + k \int_S^T \int_{\Omega} u(x, t) u_t(x, t - \tau) dx dt \\
& - \tilde{\mu} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \int_S^T \int_{\Omega} \int_0^{\infty} \mu(s) \nabla u(x, t) \nabla \eta^t(x, s) ds dx dt = 0,
\end{aligned}$$

where we used (2.1).



Then,

$$\begin{aligned}
(1 - \tilde{\mu}) & \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt \\
&= \int_S^T \int_{\Omega} u_t^2(x, t) dx dt - \left[ \int_{\Omega} u(x, t) u_t(x, t) dx \right]_S^T \\
&\quad - \theta |k| e^\tau \int_S^T \int_{\Omega} u(x, t) u_t(x, t) dx dt - k \int_S^T \int_{\Omega} u(x, t) u_t(x, t - \tau) dx dt \\
&\quad - \int_S^T \int_{\Omega} \int_0^\infty \mu(s) \nabla u(x, t) \nabla \eta^t(x, s) ds dx dt.
\end{aligned} \tag{3.13}$$

In order to estimate the integral

$$\int_S^T \left| \int_{\Omega} \int_0^\infty \mu(s) \nabla \eta^t(x, s) \nabla u(x, t) ds dx \right| dt,$$

we note that, for all  $\varepsilon > 0$ ,

$$\begin{aligned}
& \int_S^T \left( \int_{\Omega} |\nabla u(x, t)|^2 dx \right)^{1/2} \int_0^\infty \mu(s) \left( \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx \right)^{1/2} ds dt \\
& \leq \frac{\varepsilon}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \frac{1}{2\varepsilon} \int_S^T \left[ \int_0^\infty \mu(s) \left( \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx \right)^{1/2} ds \right]^2 dt.
\end{aligned} \tag{3.14}$$

We have

$$\begin{aligned}
& \int_S^T \left[ \int_0^\infty \mu(s) \left( \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx \right)^{1/2} ds \right]^2 dt \\
& \leq \int_S^T \left( \int_0^\infty \mu(s) ds \right) \left( \int_0^\infty \mu(s) \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx ds \right) dt \\
& = \tilde{\mu} \int_S^T \int_0^\infty \mu(s) \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx ds dt.
\end{aligned}$$

Therefore, recalling the estimate (3.6), we obtain

$$\int_S^T \left[ \int_0^\infty \mu(s) \left( \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx \right)^{1/2} ds \right]^2 dt \leq \frac{2\tilde{\mu}}{\alpha} F(S). \tag{3.15}$$

Then, (3.14) and (3.15) give

$$\begin{aligned}
& \int_S^T \left| \int_{\Omega} \int_0^\infty \mu(s) (\nabla u(x, t - s) - \nabla u(x, t)) \cdot \nabla u(x, t) ds dx \right| dt \\
& \leq \frac{\varepsilon}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \frac{\tilde{\mu}}{\alpha \varepsilon} F(S).
\end{aligned} \tag{3.16}$$

Now observe that

$$F(t) \geq \frac{1}{2} \int_{\Omega} u_t^2(x, t) dx + \frac{1 - \tilde{\mu}}{2} \int_{\Omega} |\nabla u(x, t)|^2 dx. \tag{3.17}$$

Then, from (3.17),

$$\frac{1}{2} \int_{\Omega} |\nabla u(x, t)|^2 dx \leq \frac{F(t)}{1 - \tilde{\mu}}, \quad (3.18)$$

and also, from Poincaré's inequality,

$$\frac{1}{2} \int_{\Omega} |u(x, t)|^2 dx \leq \frac{C_P}{2} \int_{\Omega} |\nabla u(x, t)|^2 dx \leq \frac{C_P}{1 - \tilde{\mu}} F(t). \quad (3.19)$$

Using the above inequalities

$$\left| \int_{\Omega} u_t(x, t) u(x, t) dx \right| \leq \frac{1}{2} \int_{\Omega} u_t^2(x, t) dx + \frac{1}{2} \int_{\Omega} u^2(x, t) dx \leq F(t) \left( 1 + \frac{C_P}{1 - \tilde{\mu}} \right). \quad (3.20)$$

Therefore,

$$- \left[ \int_{\Omega} u_t(x, t) u(x, t) dx \right]_S^T \leq 2F(S) \left( 1 + \frac{C_P}{1 - \tilde{\mu}} \right), \quad (3.21)$$

where we used also the fact that  $F$  is decreasing. Using (3.16), (3.21) and Cauchy-Schwarz's inequality in order to bound the terms in the right-hand side of (3.13) we have that for any  $\varepsilon > 0$ ,

$$\begin{aligned} (1 - \tilde{\mu}) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt &\leq \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + \frac{\varepsilon}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt \\ &+ \frac{\tilde{\mu}}{\alpha \varepsilon} F(S) + 2 \left( 1 + \frac{C_P}{1 - \tilde{\mu}} \right) F(S) + \frac{\theta |k| e^{\tau}}{2} \int_S^T \int_{\Omega} u^2(x, t) dx dt \\ &+ \frac{\theta |k| e^{\tau}}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + \frac{|k|}{2} \int_S^T \int_{\Omega} u^2(x, t) dx dt + \frac{|k|}{2} \int_S^T \int_{\Omega} u_t^2(x, t - \tau) dx dt. \end{aligned}$$

Therefore, from Poincaré's inequality,

$$\begin{aligned} (1 - \tilde{\mu}) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt &\leq \left( 1 + \frac{\theta |k| e^{\tau}}{2} \right) \int_S^T \int_{\Omega} u_t^2(x, t) dx dt \\ &+ \frac{\varepsilon + (\theta e^{\tau} + 1) |k| C_P}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \frac{\tilde{\mu}}{\alpha \varepsilon} F(S) \\ &+ 2 \left( 1 + \frac{C_P}{1 - \tilde{\mu}} \right) F(S) + \frac{|k|}{2} \int_S^T \int_{\Omega} u_t^2(x, t - \tau) dx dt. \end{aligned}$$

Now, observe that from (3.4),

$$\begin{aligned} \frac{|k|}{2} \int_S^T \int_{\Omega} u_t^2(x, t - \tau) dx dt &= \frac{1}{\theta - 1} \frac{|k|(\theta - 1)}{2} \int_S^T \int_{\Omega} u_t^2(x, t - \tau) dx dt \\ &\leq \frac{1}{\theta - 1} \int_S^T (-F'(t)) dt \leq \frac{1}{\theta - 1} F(S). \end{aligned} \quad (3.22)$$

Now, choose  $\varepsilon = \frac{1 - \tilde{\mu}}{2}$ . Thus, using (3.10) and also (3.22) we obtain

$$\begin{aligned} (1 - \tilde{\mu}) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt &\leq 2 \left( 1 + \frac{\theta |k| e^{\tau}}{2} \right) \int_S^T \int_{\Omega} u_t^2(x, t) dx dt \\ &+ 4 \left( 1 + \frac{\tilde{\mu}}{\alpha(1 - \tilde{\mu})} + \frac{C_P}{1 - \tilde{\mu}} + \frac{1}{2(\theta - 1)} \right) F(S), \end{aligned}$$

that is (3.11) with constants  $C_0, C_1$  given by (3.12). ■

**Lemma 3.6** *For any  $T \geq S \geq 0$ , the following identity holds:*

$$\begin{aligned}
\tilde{\mu} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt &= \left[ \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx \right]_S^T \\
&\quad - \int_S^T \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu'(s) \eta^t(x, s) ds dx dt \\
&\quad + (1 - \tilde{\mu}) \int_S^T \int_{\Omega} \nabla u(x, t) \int_0^{\infty} \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&\quad + \int_S^T \int_{\Omega} \left| \int_0^{\infty} \mu(s) \nabla \eta^t(x, s) ds \right|^2 dx dt \\
&\quad + \theta |k| e^{\tau} \int_S^T \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx dt \\
&\quad + k \int_S^T \int_{\Omega} u_t(x, t - \tau) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx dt.
\end{aligned} \tag{3.23}$$

**Proof.** We multiply equation (3.1) by  $\int_0^{\infty} \mu(s) \eta^t(x, s) ds$  and integrate by parts on  $[S, T] \times \Omega$ . We obtain

$$\begin{aligned}
\int_S^T \int_{\Omega} \left\{ u_{tt}(x, t) - \Delta u(x, t) + \int_0^{\infty} \mu(s) \Delta u(x, t - s) ds + k u_t(x, t - \tau) + \theta |k| e^{\tau} u_t(x, t) \right\} \\
\times \left\{ \int_0^{\infty} \mu(s) \eta^t(x, s) ds \right\} dx dt = 0.
\end{aligned} \tag{3.24}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_S^T \int_{\Omega} u_{tt}(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx dt \\
&= \left[ \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx \right]_S^T \\
&\quad - \int_S^T \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) (u_t(x, t) - \eta_s^t(x, s)) ds dx dt \\
&= \left[ \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx \right]_S^T \\
&\quad - \tilde{\mu} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt - \int_S^T \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu'(s) \eta^t(x, s) ds dx dt.
\end{aligned} \tag{3.25}$$

Moreover,

$$\begin{aligned}
& \int_S^T \int_\Omega \left( -\Delta u(x, t) + \int_0^\infty \mu(s) \Delta u(x, t-s) ds \right) \int_0^\infty \mu(s) \eta^t(x, s) ds dx dt \\
&= \int_S^T \int_\Omega \nabla u(x, t) \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&\quad - \int_S^T \int_\Omega \int_0^\infty \mu(s) \nabla u(x, t-s) ds \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&= \int_S^T \int_\Omega \nabla u(x, t) \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&\quad + \int_S^T \int_\Omega \int_0^\infty \mu(s) (\nabla u(x, t) - \nabla u(x, t-s)) ds \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&\quad - \tilde{\mu} \int_S^T \int_\Omega \nabla u(x, t) \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&= (1 - \tilde{\mu}) \int_S^T \int_\Omega \nabla u(x, t) \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
&\quad + \int_S^T \int_\Omega \left| \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds \right|^2 dx dt.
\end{aligned} \tag{3.26}$$

Using (3.25) and (3.26) in (3.24) we obtain (3.23). ■

**Lemma 3.7** *Assume*

$$|k| < \frac{\tilde{\mu}}{2\theta} e^{-\tau}. \tag{3.27}$$

*Then, for any  $T \geq S > 0$  and for any  $\varepsilon > 0$  we have*

$$\int_S^T \int_\Omega u_t^2(x, t) dx dt \leq \varepsilon \int_S^T \int_\Omega |\nabla u(x, t)|^2 dx dt + C_2 F(S), \tag{3.28}$$

*where the constant  $C_2 := C_2(\varepsilon)$  is defined by*

$$C_2 = \frac{4}{\tilde{\mu}} \left( 1 + \frac{1}{2\theta - 1} + \frac{\mu(0)}{\tilde{\mu}} C_P \right) + 4C_P + \frac{2}{\alpha} \left( 2 + \frac{(1 - \tilde{\mu})^2}{\tilde{\mu}\varepsilon} + C_P |k| (\theta e^\tau + 1) \right). \tag{3.29}$$

**Proof.** In order to prove Lemma 3.7 we have to estimate the terms of the right-hand side of (3.23). First we have,

$$\begin{aligned}
& \left| \int_\Omega u_t(x, t) \int_0^\infty \mu(s) \eta^t(x, s) ds dx \right| \\
&\leq \int_0^\infty \mu(s) \left( \int_\Omega |u_t(x, t)| |\eta^t(x, s)| dx \right) ds \\
&\leq \int_0^\infty \mu(s) \left( \int_\Omega u_t^2(x, t) dx \right)^{1/2} \left( \int_\Omega (\eta^t(x, s))^2 dx \right)^{1/2} ds \\
&\leq \frac{1}{2} \int_\Omega u_t^2(x, t) dx + \frac{1}{2} \left( \int_0^\infty \mu(s) \left( \int_\Omega (\eta^t(x, s))^2 dx \right)^{1/2} ds \right)^2.
\end{aligned}$$

Then, recalling (2.17) and using Hölder's inequality, we deduce

$$\begin{aligned}
& \left| \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx \right| \\
& \leq F(t) + \frac{C_P}{2} \left( \int_0^{\infty} \mu(s) \left( \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx \right)^{1/2} ds \right)^2 \\
& \leq F(t) + \frac{C_P}{2} \tilde{\mu} \int_0^{\infty} \mu(s) \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx ds \leq F(t) (1 + C_P \tilde{\mu}).
\end{aligned} \tag{3.30}$$

Therefore,

$$\left[ \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu(s) \eta^t(x, s) ds dx \right]_S^T \leq 2(1 + C_P \tilde{\mu}) F(S). \tag{3.31}$$

Now we proceed to estimate the second term in the right-hand side of (3.23). For any  $\delta > 0$  we have

$$\begin{aligned}
& \left| \int_S^T \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu'(s) \eta^t(x, s) ds dx dt \right| \\
& \leq \int_S^T \left( \int_{\Omega} u_t^2(x, t) dx \right)^{1/2} \left( \int_{\Omega} \left( \int_0^{\infty} \mu'(s) \eta^t(x, s) ds \right)^2 dx \right)^{1/2} dt \\
& \leq \frac{\delta}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + \frac{1}{2\delta} \int_S^T \int_{\Omega} \left( \int_0^{\infty} \mu'(s) \eta^t(x, s) ds \right)^2 dx dt \\
& \leq \frac{\delta}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt \\
& \quad + \frac{1}{2\delta} \int_S^T \int_{\Omega} \int_0^{\infty} (-\mu'(s)) ds \int_0^{\infty} |\mu'(s)| (\eta^t(x, s))^2 ds dx dt,
\end{aligned}$$

and then by Corollary 3.3

$$\begin{aligned}
& \left| \int_S^T \int_{\Omega} u_t(x, t) \int_0^{\infty} \mu'(s) \eta^t(x, s) ds dx dt \right| \\
& \leq \frac{\delta}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt - \frac{\mu(0)}{2\delta} C_P \int_S^T \int_0^{\infty} \mu'(s) \int_{\Omega} |\nabla \eta^t(x, s)|^2 dx ds dt \\
& \leq \frac{\delta}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt + \frac{\mu(0)}{\delta} C_P F(S).
\end{aligned} \tag{3.32}$$

Moreover, by (3.6) we have

$$\begin{aligned}
& \int_S^T \int_{\Omega} \left| \int_0^{\infty} \mu(s) \nabla \eta^t(x, s) ds \right|^2 dx dt \\
& \leq \int_S^T \int_{\Omega} \tilde{\mu} \int_0^{\infty} \mu(s) |\nabla \eta^t(x, s)|^2 ds dx dt \\
& \leq \frac{2\tilde{\mu}}{\alpha} F(S).
\end{aligned} \tag{3.33}$$

Then, it results also

$$\begin{aligned}
& \int_S^T \int_\Omega \nabla u(x, t) \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx dt \\
& \leq \int_S^T \left| \int_\Omega \nabla u(x, t) \int_0^\infty \mu(s) \nabla \eta^t(x, s) ds dx \right| dt \\
& \leq \frac{\varepsilon}{2} \int_S^T \int_\Omega |\nabla u(x, t)|^2 dx dt + \frac{\tilde{\mu}}{\alpha \varepsilon} F(S).
\end{aligned} \tag{3.34}$$

Now we estimate the last two integrals in the right-hand side of (3.23).

$$\begin{aligned}
& \theta |k| e^\tau \int_S^T \int_\Omega u_t(x, t) \int_0^\infty \mu(s) \eta^t(x, s) ds dx dt \\
& + k \int_S^T \int_\Omega u_t(x, t - \tau) \int_0^\infty \mu(s) \eta^t(x, s) ds dx dt \\
& \leq \frac{|k|}{2} \int_S^T \int_\Omega u_t^2(x, t - \tau) dx dt + \frac{\theta |k| e^\tau}{2} \int_S^T \int_\Omega u_t^2(x, t) dx dt \\
& \quad + \frac{|k|(1 + \theta e^\tau)}{2} \int_S^T \int_\Omega \left( \int_0^\infty \mu(s) \eta^t(x, s) ds \right)^2 dx dt \\
& \leq \frac{|k|}{2} \int_S^T \int_\Omega u_t^2(x, t - \tau) dx dt + \frac{\theta |k| e^\tau}{2} \int_S^T \int_\Omega u_t^2(x, t) dx dt \\
& \quad + \frac{|k|(1 + \theta e^\tau)}{2} C_P \tilde{\mu} \int_S^T \int_\Omega \int_0^\infty \mu(s) |\nabla \eta^t(x, s)|^2 ds dx dt.
\end{aligned}$$

Therefore, recalling (3.6) and (3.22), we have

$$\begin{aligned}
& \theta |k| e^\tau \int_S^T \int_\Omega u_t(x, t) \int_0^\infty \mu(s) \eta^t(x, s) ds dx dt \\
& + k \int_S^T \int_\Omega u_t(x, t - \tau) \int_0^\infty \mu(s) \eta^t(x, s) ds dx dt \\
& \leq \frac{|k|}{2} \int_S^T \int_\Omega u_t^2(x, t - \tau) dx dt + \frac{\theta |k| e^\tau}{2} \int_S^T \int_\Omega u_t^2(x, t) dx dt + C_P (|k|(\theta e^\tau + 1)) \frac{\tilde{\mu}}{\alpha} F(S) \\
& \leq \frac{1}{\theta - 1} F(S) + \frac{\theta |k| e^\tau}{2} \int_S^T \int_\Omega u_t^2(x, t) dx dt + \frac{C_P \tilde{\mu}}{\alpha} |k|(\theta e^\tau + 1) F(S).
\end{aligned} \tag{3.35}$$

Using (3.31)–(3.35) in (3.23) we obtain

$$\begin{aligned}
& \left( \tilde{\mu} - \frac{\theta |k| e^\tau}{2} - \frac{\delta}{2} \right) \int_S^T \int_\Omega u_t^2(x, t) dx \leq \frac{\varepsilon}{2} (1 - \tilde{\mu}) \int_S^T \int_\Omega |\nabla u(x, t)|^2 dx dt \\
& \quad + \frac{1}{\theta - 1} F(S) + 2(1 + C_P \tilde{\mu}) F(S) + \frac{\mu(0)}{\delta} C_P F(S) \\
& \quad + \frac{\tilde{\mu}}{\alpha} \left( \frac{1 - \tilde{\mu}}{\varepsilon} + 2 \right) F(S) + C_P \tilde{\mu} \frac{|k|(\theta e^\tau + 1)}{\alpha} F(S).
\end{aligned} \tag{3.36}$$

Now, fix  $\delta = \frac{\tilde{\mu}}{2}$ . Then, from (3.27), for any  $T \geq S > 0$ , we have

$$\begin{aligned} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt &\leq \frac{\varepsilon}{\tilde{\mu}} (1 - \tilde{\mu}) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt \\ &\quad + \frac{2}{\tilde{\mu}} \left( 2(1 + C_P \tilde{\mu}) + \frac{1}{\theta - 1} + 2 \frac{\mu(0)}{\tilde{\mu}} C_P + \frac{\tilde{\mu}}{\alpha} \left( 2 + \frac{1 - \tilde{\mu}}{\varepsilon} + C_P |k| (\theta e^{\tau} + 1) \right) \right) F(S), \end{aligned} \quad (3.37)$$

that is (3.28) with constant  $C_2$  as in (3.29). ■

**Lemma 3.8** *Assume*

$$|k| < \min \left\{ \frac{1 - \tilde{\mu}}{2C_P(\theta e^{\tau} + 1)}, \frac{\tilde{\mu}}{2\theta} e^{-\tau} \right\}. \quad (3.38)$$

Then, for any  $T \geq S > 0$ ,

$$\frac{1 - \tilde{\mu}}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \frac{1}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt \leq C^* F(S), \quad (3.39)$$

with

$$C^* = C_0 C_2 + C_1 + C_2, \quad (3.40)$$

where  $C_0$  and  $C_1$  are the constants defined by (3.12) and

$$\begin{aligned} C_2 := C_2 \left( \frac{1 - \tilde{\mu}}{2(C_0 + 1)} \right) &= \frac{4}{\tilde{\mu}} \left( 1 + \frac{1}{2\theta - 1} + \frac{\mu(0)}{\tilde{\mu}} C_P \right) + 4C_P \\ &\quad + \frac{2}{\alpha} \left( 2 + (6 + 2\theta |k| e^{\tau}) \frac{1 - \tilde{\mu}}{\tilde{\mu}} + C_P |k| (\theta e^{\tau} + 1) \right). \end{aligned} \quad (3.41)$$

**Proof.** The assumptions of previous lemmas are verified. Thus, we can use (3.28) in (3.11). Then,

$$\begin{aligned} (1 - \tilde{\mu}) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx &\leq C_0 \varepsilon \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + (C_0 C_2 + C_1) F(S). \end{aligned} \quad (3.42)$$

Therefore, from (3.28) and (3.42), we obtain

$$\begin{aligned} \frac{1 - \tilde{\mu}}{2} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx &+ \frac{1}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt \\ &\leq \frac{\varepsilon}{2} (C_0 + 1) \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx dt + \frac{1}{2} (C_0 C_2 + C_1 + C_2) F(S). \end{aligned} \quad (3.43)$$

Now, fix

$$\varepsilon = \frac{1 - \tilde{\mu}}{2(C_0 + 1)}.$$

Then, from (3.43) we deduce

$$\begin{aligned} & \frac{1 - \tilde{\mu}}{4} \int_S^T \int_{\Omega} |\nabla u(x, t)|^2 dx + \frac{1}{2} \int_S^T \int_{\Omega} u_t^2(x, t) dx dt \\ & \leq \frac{1}{2} (C_0 C_2 + C_1 + C_2) F(S), \end{aligned}$$

where, from (3.29) with the above choice of  $\varepsilon$ ,  $C_2$  is as in (3.41). This clearly implies (3.39) with  $C^*$  as in (3.40). ■

**Proof of Theorem 3.4.** Notice also that (3.5) directly implies that

$$\frac{\theta|k|e^\tau}{2} \int_S^T \int_{t-\tau}^t e^{-(t-s)} \int_{\Omega} u_t^2(x, s) dx ds dt \leq - \int_S^T F'(t) dt \leq F(S). \quad (3.44)$$

Let us define  $\bar{k}$  as

$$\bar{k} := \min \left\{ \frac{1 - \tilde{\mu}}{2C_P(\theta e^\tau + 1)}, \frac{\tilde{\mu}}{2\theta} e^{-\tau} \right\}. \quad (3.45)$$

Then, if  $|k| < \bar{k}$ , using (3.39), (3.6) and (3.44) in (3.8), we obtain

$$\int_S^T F(t) dt \leq C^* F(S) + \frac{1}{\alpha} F(S) + F(S).$$

Therefore (3.7) is verified with

$$C = C^* + 1 + \frac{1}{\alpha}, \quad (3.46)$$

where  $C^*$  is as in (3.40) with  $C_0, C_1$  and  $C_2$  defined in (3.12) and (3.41). ■

**Proof of Theorem 2.2** From Theorem 3.4 and Lemma 2.4, it follows that for any solution of the auxiliary problem (3.1) – (3.3) if  $|k| < \bar{k}$ , we have

$$F(t) \leq F(0) e^{1 - \tilde{\sigma} t}, \quad t \geq 0, \quad (3.47)$$

with

$$\tilde{\sigma} := \frac{1}{C}, \quad (3.48)$$

where  $C$  is as in (3.46).

From this and Theorem 2.3 we deduce that Theorem 2.2 holds, with  $\sigma := \tilde{\sigma} - e\theta|k|e^\tau$ , if

$$-\tilde{\sigma} + e\theta|k|e^\tau < 0,$$

that is if the delay parameter  $k$  satisfies

$$|k| < g(|k|) := \frac{1}{C e \theta e^\tau}, \quad (3.49)$$

with  $C := C(|k|)$  defined in (3.46). Now observe that (3.49) is satisfied for  $k = 0$  because  $g(0) > 0$ . Moreover, by recalling the definitions of the constants  $C_0, C_1, C_2$  and  $C^*$ , used to define  $C$ , we note that  $g : [0, \infty) \rightarrow (0, \infty)$  is a continuous decreasing function satisfying

$$g(|k|) \rightarrow 0 \quad \text{for} \quad |k| \rightarrow \infty.$$



Thus, there exists a unique constant  $\hat{k} > 0$  such that  $\hat{k} = g(\hat{k})$ . We can then conclude that for any  $\theta$  in the definition (2.17) of the energy  $F(\cdot)$ , inequality (3.49) is satisfied for every  $k$  with

$$|k| < k_0 = \min\{\hat{k}, \bar{k}\}. \quad \blacksquare$$

**Remark 3.9** We can compute an explicit lower bound for  $k_0$ . Indeed (3.49) may be rewritten as

$$|k|\theta e^{\tau+1} \left( C^* + 1 + \frac{1}{\alpha} \right) < 1.$$

Then, from (3.40), we have

$$[1 + 1/\alpha + C_2(C_0 + 1) + C_1]\theta e^{\tau+1}|k| < 1, \quad (3.50)$$

that is

$$\begin{aligned} h(|k|) := & \left\{ 1 + \frac{1}{\alpha} + \left[ \frac{4}{\tilde{\mu}} \left( 1 + \frac{1}{2} \frac{1}{\theta - 1} + \frac{\mu(0)}{\tilde{\mu}} C_P \right) + 4C_P \right. \right. \\ & \left. \left. + \frac{2}{\alpha} \left( 2 + (6 + 2\theta|k|e^\tau) \frac{1 - \tilde{\mu}}{\tilde{\mu}} + C_P|k|(\theta e^\tau + 1) \right) \right] (3 + \theta|k|e^\tau) \right. \\ & \left. + 4 \left( 1 + \frac{\tilde{\mu}}{\alpha(1 - \tilde{\mu})} + \frac{C_P}{1 - \tilde{\mu}} + \frac{1}{2(\theta - 1)} \right) \right\} \theta e^{\tau+1}|k| < 1. \end{aligned} \quad (3.51)$$

Now, we use the assumption  $|k| < \bar{k}$  with  $\bar{k}$  defined in (3.45) in order to majorize the left-hand side of (3.51),  $h(|k|)$ , with a linear function. We have

$$\begin{aligned} h(|k|) \leq & \left\{ 1 + \frac{1}{\alpha} + \left[ \frac{4}{\tilde{\mu}} \left( 1 + \frac{1}{2} \frac{1}{\theta - 1} + \frac{\mu(0)}{\tilde{\mu}} C_P \right) + 4C_P \right. \right. \\ & \left. \left. + \frac{2}{\alpha} \left( 2 + (6 + \tilde{\mu}) \frac{1 - \tilde{\mu}}{\tilde{\mu}} + \frac{1 - \tilde{\mu}}{2} \right) \right] (3 + \tilde{\mu}/2) \right. \\ & \left. + 4 \left( 1 + \frac{\tilde{\mu}}{\alpha(1 - \tilde{\mu})} + \frac{C_P}{1 - \tilde{\mu}} + \frac{1}{2(\theta - 1)} \right) \right\} \theta e^{\tau+1}|k|, \end{aligned} \quad (3.52)$$

from which follows

$$h(|k|) \leq \left( 1 + \frac{1}{\alpha} \gamma_1 + \gamma_2 \right) \theta |k| e^{\tau+1},$$

with

$$\begin{aligned} \gamma_1 &= \gamma_1(\tilde{\mu}) = 4 \frac{\tilde{\mu}}{1 - \tilde{\mu}} - 8 + \frac{36}{\tilde{\mu}} - \frac{23}{2} \tilde{\mu} - \frac{3}{2} \tilde{\mu}^2, \\ \gamma_2 &= \gamma_2(\mu(0), \tilde{\mu}, \theta, C_P) \\ &= 6 + 12C_P + \frac{3}{\theta - 1} + \frac{12}{\tilde{\mu}} + \frac{6}{\tilde{\mu}(\theta - 1)} \\ &\quad + 12 \frac{\mu(0)}{\tilde{\mu}^2} C_P + 2 \frac{\mu(0)}{\tilde{\mu}} C_P + 2C_P \tilde{\mu} + \frac{4C_P}{1 - \tilde{\mu}}. \end{aligned}$$

Then, we deduce the following explicit lower bound

$$k_0 \geq \frac{e^{-(\tau+1)}}{\theta(1 + \frac{1}{\alpha}\gamma_1 + \gamma_2)}, \quad (3.53)$$

with  $\gamma_1, \gamma_2$  as before. For example, if we take

$$\mu(t) = e^{-2t},$$

then  $\tilde{\mu} = 1/2$  and so, fixing  $\theta = 2$ , we can compute  $\gamma_1 = \frac{495}{8}$ ,  $\gamma_2 = 45 + 73C_P$ . Hence, for this particular choice of the memory kernel, we obtain

$$k_0 \geq \frac{8e^{-(\tau+1)}}{1231 + 1168C_P}.$$

**Remark 3.10** In the case  $\tau = 0$  and  $k < 0$ , namely viscoelastic wave equation with anti-damping, we can simplify previous arguments. Indeed, the absence of time delay allows us to take  $\theta = 1$  obtaining an exponential stability estimate under the condition

$$|k| < \left(C_1 + 3C_2 + \frac{1}{\alpha}\right)^{-1} \frac{1}{e},$$

where

$$C_1 = 4\left(1 + \frac{\tilde{\mu}}{\alpha(1 - \tilde{\mu})} + \frac{C_P}{1 - \tilde{\mu}}\right)$$

and

$$C_2 = \frac{2}{\tilde{\mu}}\left(2 + \frac{\mu(0)}{\tilde{\mu}}C_P\right) + 4C_P + \frac{2}{\alpha}\left(2 + 6\frac{1 - \tilde{\mu}}{\tilde{\mu}}\right).$$

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